

D-wave superconductivity in doped Mott insulators

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(Dated: Dec 21, 2001)

The effect of proximity to a Mott insulating phase on the charge transport properties of a superconductor is determined. An action describing the low energy physics is formulated and different scenarios for the approach to the Mott phase are distinguished by different variation with doping of the parameters in the action. A crucial issue is found to be the doping dependence of the *quasiparticle charge* which is defined here and which controls the temperature and field dependence of the electromagnetic response functions. Presently available data on high- T_c superconductors are analysed. The data, while neither complete nor entirely consistent, suggest that neither the quasiparticle velocity nor the quasiparticle charge vanish as the Mott phase is approached, in contradiction to the predictions of several widely studied theories of lightly doped Mott insulators. Implications of the results for the structure of vortices in high- T_c superconductors are determined.

I. INTRODUCTION

High- T_c superconductors are created by doping an antiferromagnetic 'Mott insulating' parent material¹. A Mott insulator is a material which band theory would predict to be metallic but which, because of electron-electron interactions, is insulating². It is conventional to distinguish the different doping levels as 'overdoped', 'optimally doped', and 'underdoped'. Optimal doping is generally defined as the carrier concentration which maximizes the resistively-defined superconducting transition temperature T_c . Overdoped materials have more added carriers and underdoped materials have fewer. All high- T_c materials exhibit behavior which deviates from the 'Fermi-liquid-Migdal-Eliashberg' theory which describes most metals; however, the deviations grow more pronounced as the doping is reduced towards the Mott insulator, and indeed understanding the physics of the underdoped regime (in other words, of the approach to the Mott insulating phase) has emerged as one of the crucial questions in the high- T_c field.

High- T_c superconductivity remains a controversial subject¹. However, there is general agreement that one important phase, which may actually be observed in sufficiently clean materials, is a homogeneous superconducting phase characterized by an energy gap vanishing only along the nodal directions ($p_x = \pm p_y$ in a material with tetragonal symmetry) and possessing conventional quasiparticle and supercurrent excitations. The low temperature behavior of a d-wave superconductor is described by a general action, which depends on four parameters which are defined below. The behavior of the parameters as the Mott phase is approached is seen to reveal information about the underlying physics of the Mott transition and governs the structure of vortices and thereby the doping dependence of such macroscopic quantities as the upper critical field. In this paper (which is largely a review) we describe the information which may in principle be obtained from the low T properties, summarize the present status of the experimental values of the different parameters, and (although the presently available data are neither fully consistent nor complete) outline the apparent physical implications of the results. Two subsequent papers are planned: one presenting the derivation of the general low temperature action from different microscopic theories and one using it to analyse the vortex properties (in particular $H_{c2}(T)$ and paraconductivity) in more detail.

Specifically, in this paper we consider a homogeneous d-wave superconducting phase, assuming in particular that there is no spontaneously broken time reversal symmetry and there are no other relevant excitations apart from the quasiparticle and phase fluctuations. (We also restrict attention to two dimensional models, but this restriction can easily be lifted if desired). We find that a crucial parameter is what we define as the effective charge of the quasiparticles. This can be determined at low T by relating the observed electronic specific heat and photoemission (which essentially measure the number and velocity of quasiparticles) to the temperature dependence of the London penetration depth (which is related to the ability of these states to carry an electrical current). Different scenarios for the approach to the Mott transition produce striking differences in the variation, with doping, of this charge. In particular, theories involving some form of spin charge separation seem to lead to a vanishing of the quasiparticle charge as the Mott phase is approached. We analyze available data to determine which scenario actually occurs.

One very important feature of a superconductor is the structure of vortices introduced by thermal fluctuations or via a magnetic field. A vortex is characterized by a core, which may be defined in two ways: either via the density of quasiparticle states, which is higher near the core and drops as one moves away from the core region, or via the supercurrent, which varies as $1/r$ far from the core and drops to zero inside the core. In several of the theoretical

approaches to the physics of the lightly doped Mott insulator (including the one which seems to best fit the data discussed below), the core size *as defined from the supercurrent* may become very large. This, combined with the behavior of the quasiparticle charge, has remarkable implications for the size of the critical region, for the behavior of the upper critical field H_{c2} and for the physics of the superconductor-insulator transition which must occur as the doping goes to zero. These implications were pointed out in³ and will be further analysed by us in a subsequent paper.

The rest of this paper is organized as follows. In section II we present the low energy, long wavelength theory of a d-wave superconductor near a Mott insulator. In Section III we discuss the available experimental evidence concerning the value and doping dependence of the parameters of the theory. Section IV presents the limits of validity of the low energy action we discuss, along with discussion of the physics occurring when these limits are exceeded. A conclusion discusses the physical implications of our formalism and findings. The Appendix gives the details of calculations of the field dependence of the specific heat and superfluid stiffness in the vortex state.

II. LOW ENERGY THEORY

At low temperatures the state of all (even underdoped) superconducting cuprates seems to be a conventional $d_{x^2-y^2}$ superconductor^{4,5,6}: it is described by the usual low energy degrees of freedom, namely a superconducting phase variable $\phi(r, t)$ and, as will be discussed in more detail below, apparently conventional fermionic quasiparticle excitations⁷. Gradients of the phase correspond to supercurrents. Longitudinal supercurrents lead to charge fluctuations which are coupled by the long-ranged Coulomb interaction. Transverse supercurrents are described by a *phase stiffness* ρ_s whose long-wavelength limit may be deduced from measurements of the London penetration depth. In high- T_c materials, the low- T limit of ρ_s is strongly x -dependent, vanishing roughly linearly as $x \rightarrow 0$. This behavior is understood as a consequence of the suppression of charge fluctuations as the Mott insulating phase is approached, and appears to be related to the decrease of the resistively defined T_c as $x \rightarrow 0$. Indeed in the very early days of high- T_c Uemura and co-workers showed that in underdoped materials the ratio $T_c(x)/\rho_s(T \rightarrow 0, x)$ was essentially x -independent⁸. At roughly the same time this behavior was shown by a number of workers to follow naturally from theoretical models of superconductivity near a Mott transition⁹, and later Emery and Kivelson argued that the behavior could be understood in a more model-independent way as a consequence of the unusually small phase stiffness characteristic of high- T_c materials¹⁰. Recently Corson et. al reported direct evidence that in underdoped $Bi_2Sr_2Cu_2O_8$ the superconducting transition was indeed of the vortex-unbinding type driven by a small phase stiffness¹¹.

At scales less than maximal value of the gap, Δ_0 , the physics of a two dimensional superconductor with tetragonal symmetry and a $d_{x^2-y^2}$ gap function is described by an effective Lagrangian density \mathcal{L} involving the phase ϕ of the superconductor and quasiparticles excited out of the superconducting condensate:

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_F + \mathcal{L}_{mix} \quad (1)$$

Here $\mathcal{L}_F = \partial_\tau - H_D$ is the usual Dirac action describing the 'nodal quasiparticles' excited in the vicinity of the nodes of the d-wave gap function. In a superconductor, the fermionic energy spectrum is given by $E_p = \pm \sqrt{\varepsilon_p^2 + \Delta_p^2}$ with ε_p the energy spectrum of the underlying fermions and Δ_p the gap function. For a $d_{x^2-y^2}$ gap function, Δ_p vanishes linearly in the four nodes, i.e. for $\vec{p} \parallel (\pm\pi, \pm\pi)$. It is convenient to measure momentum from the fermi point in the nodal direction and to parametrize the fermion dispersion by two velocities: one, v_F , of the order of the underlying fermi velocity describing motion perpendicular to the direction in which the gap varies, and one, v_Δ , describing the opening of the gap and of order Δ_0/p_F , obtaining

$$E_p = \sqrt{(v_F p_1)^2 + (v_\Delta p_2)^2} \quad (2)$$

We take the fermions to be normal ordered in the basis which diagonalizes H_D , so the contribution of the negative energy (filled Dirac sea) states is subsumed in the phase Lagrangian density \mathcal{L}_ϕ which we write as

$$\mathcal{L}_\phi = \frac{1}{2} (\partial_i \phi + 2ieA_i) * \rho_{s0}^{ij}(r - r') * (\partial_j \phi - 2ieA_j) \quad (3)$$

Here i is a Cartesian direction, \mathbf{A} is the vector potential and we have allowed for non-locality in space so the $*$ represents convolution.

The quantity ρ_{s0}^{ij} is a diagonal matrix with dimension of *energy/length*². Its components $\rho_{s0}^{xx}(r) = \rho_{s0}^{yy}(r) \equiv \rho_{s0}(r)$ (we assume tetragonal symmetry) are related, in the absence of quasiparticle excitations, to the conventionally defined superfluid stiffness ρ_s (measurable, e.g. from the London penetration depth) by

$$\rho_s(T = 0, H = 0) = \int d^2r \rho_{s0}(r) \quad (4)$$

Systems near a Mott transition are characterized by a low density of mobile charges, and we therefore expect that $\rho_{S0}(r)$ has a length dependence set by this low density.

The term \mathcal{L}_{mix} gives the coupling of the phase fluctuations to the nodal quasiparticles; it may be written

$$\mathcal{L}_{mix} = \sum_{\alpha, \sigma, p, q} \left(\frac{1}{2} \partial_\mu \phi(r) + ie A_\mu(r) \right) \cdot e^{iq \cdot r} Z_p^e \vec{v}_F \psi_{p+q/2, \alpha \sigma}^+ \psi_{p-q/2, \alpha \sigma} \quad (5)$$

Here $\alpha = 1 \dots 4$ labels the four nodes of the d-wave state and Z^e is a phenomenological constant which we will show below may be thought of as the charge of a superconducting quasiparticle. It may depend on position relative to the node and on the proximity to the Mott transition and will be discussed in more detail below. It has been stated in the literature that one generically has $Z^e = 1$; but this is now known not to be correct. Eq. \mathcal{L}_{mix} is the long wavelength limit of a more general action involving also 'pairbreaking' terms such as $\psi^+ \psi^+$ with coefficients of order q .

III. PHYSICAL CONTENT

The low energy, long wavelength theory is described by four parameters: ρ_{S0} , v_F , v_Δ and the quasiparticle charge renormalization Z^e . To see how these parameters may be determined experimentally, we integrate out the fermions in the presence of static, slowly varying superflow field and vector potential, which enter via the gauge invariant combination

$$\vec{Q} = \left(\vec{\nabla} \phi(r, t) - 2ie \vec{A} \right) \quad (6)$$

and are taken to be slowly varying on the scales set by ρ_S and the fermions. We obtain for the two dimensional free energy density

$$F_{static}(\vec{Q}) = \frac{1}{2} \rho_S^0 Q^2 - 2T \sum_\alpha \int dE N(E) \ln \left[1 + \exp[-(E + \frac{1}{2} Z_p^e \vec{Q} \cdot \vec{v}_a)/T] \right] \quad (7)$$

where the 2 is for spin, the sum (α) is over the four nodes of the Dirac spectrum and we have introduced the single-node single-spin density of states per unit area

$$N(E) = \int \frac{d^2 p}{(2\pi)^2} \delta(E - E_p) = \frac{E}{2\pi v_F v_\Delta} \quad (8)$$

The specific heat may be obtained by differentiating Eq. 7 twice with respect to T and is

$$\frac{C}{T} = \frac{T}{4\pi v_F v_\Delta} \sum_\alpha \int_0^\infty dx \frac{x(x + \frac{Z_p^e \vec{Q} \cdot \vec{v}_a}{2T})^2}{\cosh^2 \left[\frac{x + \frac{Z_p^e \vec{Q} \cdot \vec{v}_a}{2T}}{2} \right]} \quad (9)$$

The integral may easily be evaluated numerically for given Q, T . Analytical results exist in the limits $Q/T \rightarrow 0$ and $Q/T \rightarrow \infty$. The zero-field specific heat coefficient (per unit area) $C(B=0, T)$ is

$$\frac{C(B=0)}{T} = \frac{18\zeta(3)T}{\pi v_F v_\Delta} \quad (10)$$

while in the high field low-T limit we obtain (after symmetrization)

$$\frac{C(Z^e v Q \gg T)}{T} = \sum_{\alpha=1 \dots 4} \frac{\pi Z^e}{12 v_F v_\Delta} \left| \vec{Q} \cdot \vec{v}_a \right| \quad (11)$$

Averaging Eq 11 over the Q -distribution characteristic of a vortex state leads to the 'Volovik' prediction¹² of a $B^{1/2}$ field dependence of the specific heat if $B_{c2} \gg B \gg B_{c1}$. In principle the result depends on the nature of the vortex state and on the relative angle between the lattice vectors characterizing the vortex lattice (if any) and the

directions corresponding to the gap nodes in the superconducting state. We have evaluated the averages and find that the dependence is weak:

$$\frac{C(B > \Phi_0 v_F^2 / T^2)}{T} = \frac{\pi Z^e}{3v_\Delta} \left(\frac{B}{\Phi_0} \right)^{1/2} A \quad (12)$$

where $A = \pi/2$ for a square vortex lattice with nodal direction aligned with the vortex lattice vector and $A = 1.52$ for 45 degrees misalignment, similarly for a triangular vortex lattice $A = 1.7$ with 5% variations as the angle is varied. Some details of the calculation are presented in the Appendix.

Similarly the differential penetration depth is given by differentiating F twice with respect to Q . For an arbitrary current distribution ρ_S is a tensor:

$$\rho_S^{ab} = \rho_{S0} \delta_{ab} - \sum_\alpha \frac{T Z^{e2} v_\alpha^a v_\alpha^b}{4\pi v_F v_\Delta} \int_0^\infty dx \frac{x}{\cosh^2 \left[x + \frac{Z^e \vec{Q} \cdot \vec{v}_\alpha}{4T} \right]} \quad (13)$$

where a, b are specific Cartesian directions and v^a is the component of v_F in the a direction. Eq. (16) describes among other things the nonlinear Meissner effect¹³: note however the importance of the charge renormalization factor Z^e .

In the weak field limit, $\rho_S^{ab} = \rho_S \delta_{ab}$ with

$$\rho_S(T) = \rho_{S0} - \frac{\ln(2) Z^{e2} v_F}{2\pi v_\Delta} T = \rho_{S0} - \frac{\ln(2) Z^{e2} v_F^2}{36\zeta(3)} \frac{C(B=0)}{T} \quad (14)$$

The factor Z^e is essentially the Landau parameter introduced in previous work¹⁴. Comparison of Eqs. 9 and 14 shows why Z^e is more appropriately interpreted as the quasiparticle charge renormalization. The usual f-sum-rule (Ferrel-Glover-Tinkham) arguments imply that the change, with temperature, in the condensate fraction is balanced by an increase in the 'normal' conductivity due to quasiparticles. Now the quasiparticle conductivity is determined by the number of carriers (which follows from the specific heat, which essentially counts excitations) and their velocity, (which may be determined from photoemission). Any remaining discrepancy with the observed $d\rho_S/dT$ must then be due to their charge, i.e. to the factor Z^e .

At large Q and low T we find the current-dependence of the superfluid stiffness to be

$$\rho_S^{ab}(Q, T=0) = \rho_{S0} \delta_{ab} - \frac{Z^{e3} v_\alpha^a v_\alpha^b}{16\pi v_F v_\Delta} \sum_\alpha \left| \vec{Q} \cdot \vec{v}_\alpha \right| \quad (15)$$

Calculations similar to those for the specific heat yield, for a vortex lattice with square or triangular symmetry, $\rho_S^{ab} = \rho_S \delta_{ab}$ with

$$\rho_S(B, T=0) = \rho_{S0} - A \frac{Z^{e3} v_F^2}{4\pi v_\Delta} \sqrt{\frac{B}{\Phi_0}} \quad (16)$$

IV. EXPERIMENTAL VALUES

A. Overview

The important parameters of the theory, ρ_S , Z^e , v_F , v_Δ may be determined from experiment. Of these, the crucial parameter is Z^e . Unfortunately, the present situation is unclear because different determinations do not agree; also most measurements determine only combinations of the fundamental quantities, so that uncertainties in one propagate into uncertainties in another. In the following sub-sections section we discuss the available data for each of the three parameters, and then in a concluding subsection summarize the results and outstanding questions.

B. ρ_s

The $T = 0$ superfluid stiffness has been reasonably well determined by muon spin rotation experiments^{8,16} and decreases as the Mott insulator is approached. The decrease is apparently roughly proportional to hole doping. We regard this result as well established and we do not discuss it further.

C. v_F

Angle-resolved photoemission measurements yield v_F ¹⁷; At present the generally accepted value for high- T_c materials (both optimal and underdoped) along the zone diagonal and in the superconducting state is^{18,19}

$$v_F = 1.8[eV - A] \quad (17)$$

The velocity apparently increases slightly as doping is decreased. There is general agreement concerning the value and doping independence of the velocity (note that even undoped materials exhibit zone diagonal velocities of approximately this magnitude). We regard this parameter as having been reasonably reliably established.

D. v_Δ

The parameter v_Δ may be obtained in three ways: from photoemission, from zero-field specific heat, and from thermal conductivity. Each method is subject to uncertainties, as outlined below.

Photoemission investigations of the form of the superconducting gap near the nodes reveal a broadened structure, with a nonvanishing density of states in a small arc around the zone diagonal²⁰. This could be an intrinsic effect, indicating a non-d-wave form of the gap function, or it could be due to pairbreaking or other sample and surface imperfections. However, evidence that the gapless arcs have a non-intrinsic origin is provided by penetration depth and thermal conductivity measurements discussed below, so we take this view here. An estimate of v_Δ from photoemission may be obtained by combining the gap maximum value Δ_0 , the standard $\cos(2\theta)$ d-wave form and the arc length from the zone diagonal fermi point to the gap maximum point, which is roughly $\pi/\sqrt{2}b$ with b the lattice constant, leading to

$$v_\Delta = \frac{2\sqrt{2}b\Delta_0}{\pi} \quad (18)$$

Estimates for the gap maximum range from $30 - 40\text{meV}$ in optimal *YBCO* (with the large values in the direction parallel to the chains and the smaller in the direction perpendicular²¹ to 40meV in *BSCCO*²² leading to

$$v_\Delta = 0.13[eV - A] \text{ (BSCCO)} \quad (19)$$

$$v_\Delta = 0.09 - 0.12[eV - A] \text{ (YBCO)} \quad (20)$$

Available photoemission evidence²² suggests that Δ_0 and therefore v_Δ if anything increase with decreasing doping; suggesting (if we interpret the maximal gap observed in the $(\pi, 0)$ direction as superconducting gap) that v_Δ increases with decreasing doping. These estimates rely on the assumption that everywhere in the zone the observed gap has a superconducting origin. While this assumption has been used by many workers, and appears to be supported by the good agreement between the simple d-wave form and the data of²², different interpretations exist in which the gap in underdoped materials has a non-superconducting origin^{23,24}.

Eq. (10) shows that measurements of the low temperature specific heat yield the product $v_F v_\Delta$. Because we regard the value of v_F as reliable, these measurements yield a value for v_Δ . In optimally doped *YBCO*, specific heat data²⁵ yield (in present notations²⁶)

$$v_F v_\Delta = 0.06 [eV - A]^2 \quad (21)$$

or

$$v_\Delta = 0.033 [eV - A] \quad (22)$$

This value is far outside the range of v_Δ suggested by photoemission. The authors of Ref.²⁵ suggest that the discrepancy occurs because there are additional contributions to the measured low-field specific heat (for example from chain states) which should not be included in the comparison between the model and data. This idea is consistent with recent microwave conductivity measurements²⁷ which find evidence for a large density of gapless excitations associated with the chains. An alternative possibility is that the gap function does not have the simple $\cos(2\theta)$ form often assumed, but instead is less strongly angle dependent near the nodes, so that v_Δ is not well estimated from the maximum gap value. Reliable measurements of the low temperature specific heat for *BSCCO* or underdoped *YBCO* are not available.

Thermal conductivity measurements yield values for v_F/v_Δ ^{7,29,30}. These results rely upon a theoretical 'universal limit' expression for the low temperature limit of a transport coefficient²⁸, and upon the assumption that this low

temperature limit has been experimentally accessed. Measurements⁷ yield $v_F/v_\Delta = 19$ for optimally doped BSCCO. For YBCO a strong doping dependence is found. As doping is decreased the ratio drops from about 19 for a presumably slightly overdoped $YBCO_{6.993}$ ³⁰ sample to 14 for a putatively optimally doped $YBCO_{6.95}$ ²⁹ to 8 for the 60 phase $YBCO_{6.73}$ ³⁰. These estimates suggest that v_Δ rapidly increases with underdoping.

$$v_\Delta = 0.095 \text{ (BSCCO, overdoped YBCO)} \quad (23)$$

$$v_\Delta = 0.13 \text{ (optimally doped YBCO)} \quad (24)$$

$$v_\Delta = 0.2 \text{ (underdoped YBCO)} \quad (25)$$

These data are roughly consistent with the v_Δ inferred from the gap maximum found in photoemission; however one should bear in mind that the increase in v_Δ found as doping is decreased corresponds to a decrease in the value of the 'universal limit' thermal conductivity. This could arise from an inhomogeneous sample (in which not all of the material is superconducting) or possibly from novel physics (not included in the basic action studied here) suppressing the ability of the quasiparticles in a doped Mott insulator to carry heat.

E. Z^e

The crucial quantity Z^e appears in combination with v_F, v_Δ and so values are subject to uncertainties, particularly in the value of v_Δ .

The temperature dependence of the penetration depth yields the combination $\frac{Z_e^2 v_F}{v_\Delta}$. In YBCO certainly and in other high- T_c materials, probably, the temperature dependence of the penetration depth in the direction transverse to the chains (if any) is only weakly material-dependent, and is linear at low T with the slope given by^{31,32}

$$\frac{d\rho_S}{d(k_B T)} \approx 0.7 \text{ (YBCO 6.6, 6.9)} \quad (26)$$

$$\approx 0.9 \text{ (BSCCO, optimal)} \quad (27)$$

Note that this linearity is inconsistent with the presence of the "gapless arcs"²⁰ in the electronic spectrum. From Eq. 14 we then obtain

$$\frac{Z_e^2 v_F}{v_\Delta} = 6 - 8 \quad (28)$$

The ability to determine Z^e by combining penetration depth data with values for v_F and v_Δ (obtained for example from thermal conductivity data) was noted by Chiao, Taillefer and co-workers⁷; the values obtained from the thermal conductivity data discussed above then yield

$$Z^e = 0.7 \text{ (Optimal BSCCO; Overdoped YBCO)} \quad (29)$$

$$Z^e = 0.8 \text{ (Optimally doped YBCO)} \quad (30)$$

$$Z^e = 1 \text{ (60K YBCO)} \quad (31)$$

The magnetic field dependence of the specific heat yields $\frac{v_\Delta}{Z_e A}$ where A is a constant (discussed above) relating to the current distribution in the vortex lattice. In optimally doped YBCO, high-field specific heat data²⁵ yield (in present notations²⁶)

$$\frac{v_\Delta}{Z_e A} = 0.09 [eV - A] \quad (32)$$

Use of our estimate $A \approx 1.7$ ²⁶ and the range quoted above for v_Δ yields

$$0.6 < Z^e < 1 \quad (33)$$

Recent microwave conductivity measurements²⁷ reveal an additional difficulty with the quantitative extraction of Z^e in YBCO: in this material the deviations from tetragonal symmetry are found to lead to a strong ($\sim 50\%$) variation in the plane conductivity (which can be separated from the chain conductivity) between electric field parallel to the chain direction and antiparallel to it. This anisotropy has not been taken into account in our analysis.

F. Summary

In summary, at present the experimental status of the parameter values characterizing the superconducting state is not completely satisfactory. The specific heat results for optimal *YBCO* suggest rather smaller v_Δ values than are found by other measurements, and photoemission and some tunnelling data suggest that v_Δ decreases as doping is reduced, while other measurements including thermal conductivity suggest that it increases. The available data suggest however that the crucial parameter Z^e is of order unity and is only weakly dependent on doping. Particularly compelling in this regard is the observed weak doping dependence of $d\rho_S/dT$, combined with the doping independence of v_F , and the indications that v_Δ increases with decreasing doping. These indications suggest that Z^e is of order unity and if anything increases as doping is decreased. Data contradicting this conclusion exist. Further experimental information would be very helpful.

V. RANGE OF APPLICABILITY OF LOW ENERGY ACTION

A. Overview

The results presented above constitute the leading temperature and Q dependence about the $T = 0, Q = 0$ limit because they are nonanalytic in the standard expansion parameters $(T/E_0)^2, (v_F Q/E_0)^2$ where E_0 is a 'microscopic' energy scale (for example the BCS gap amplitude Δ_0). We expect the expansion ceases to hold when the correction terms are of the order of the leading terms and in particular when the corrections to ρ_{S0} are of the order of ρ_{S0} .

One source of correction terms are terms of the order of Q^4 in the phase part of the action. The usual expectation from study of quantum critical points is that these become important when

$$Q \sim Q^\phi = (\rho_{S0}/E_\phi)^{1/2}/\xi_0 \quad (34)$$

$$T^\phi \sim \rho_{S0} \quad (35)$$

where E_ϕ, ξ_0 are 'microscopic' energy and length scales which do not vanish as the Mott phase or other critical point is approached.

Another correction occurs when the fermionic terms become of the order of the leading terms, i.e. when

$$T \sim T^* = \rho_{S0} \left(\frac{v_\Delta}{Z_e^2 v_F} \right) \quad (36)$$

or

$$Q \sim Q^* = \frac{2\rho_{S0}}{v_F} \left(\frac{v_\Delta}{Z_e^3 v_F} \right) \quad (37)$$

Roughly, if the fermionic terms determine the limits of validity of the low T, Q expansion, then the physics of the nonsuperconducting state is dominated by electrons and is expected to be more or less a conventional metal, whereas if the phase terms set the limits then fermions are irrelevant at the superconducting-non-superconducting critical point and the physics is presumably bosonic.

In general, the limit of validity of the low T low B expansion signals the destruction of the superconducting state. We shall discuss the superconducting non-superconducting transition on the assumption that the physics is strictly two dimensional. While this is a reasonable approximation for high- T_c materials, it is important to bear in mind that ultimately a crossover to three dimensional critical behavior will occur and that the parameter controlling the crossover is the inverse of the square of the logarithm of the superfluid stiffness anisotropy $1/\ln^2(\rho_{S\parallel}/\rho_{S\perp})$ which is not extremely small in practice, so although the two dimensional arguments provide reasonable estimates of the energy scales controlling T_c and (as discussed below) H_{c2} , a quantitative application requires some caution.

B. Thermal fluctuations, zero field

In a two dimensional material the thermally driven zero-field superconducting-non-superconducting transition is a Kosterlitz-Thouless vortex unbinding transition. It occurs at a T_{KT} satisfying $T_{KT} = 2\rho_S(T_{KT})/\pi$. Because thermal effects can only decrease ρ_S from its $T = 0$ value, the scale T^ϕ defined in Eq. 35 is an upper bound for this transition temperature.

Two kinds of thermal effects occur: fluctuations of the superconducting phase, and quasiparticle excitations. In the absence of a high density of quasiparticle excitations, longitudinal ("spin-wave") phase fluctuations involve unscreened charge fluctuations and are therefore strongly suppressed by the Coulomb interaction. In the absence of a high density of quasiparticles the only important excitations are vortex-antivortex pairs, whose energetics are governed by the scale ρ_{S0} . (In this regard we worry that the numerical studies of Ref.³³ are not quantitatively relevant to superconductors because these studies were based on the classical XY model, so a large contribution from 'spin-wave' longitudinal excitations is apparently included, whereas one would expect these to be strongly suppressed in an electronic system in which coulomb forces were important). Quasiparticle excitations will reduce $\rho_S(T)$ from its $T = 0$ value. If $\frac{v_\Delta}{Z_e^2 v_F} < 1$ then the limit T^* set by quasiparticle effects is more stringent. The physics of this limit (which seems to be favored by the data) is simply that thermal quasiparticle excitations reduce ρ_S so that the Kosterlitz-Thouless transition occurs at a temperature lower by a factor of F than one would guess from ρ_{S0} .

C. Field driven effects: low T

Application of a magnetic field $B > B_{c1}$ produces vortices in the superconducting order parameter. A superconducting vortex consists of a "core" and a "far" region. In the far region superconducting excitation spectrum is only weakly perturbed and there is a circulating supercurrent of a magnitude $j_S \rho_S / r$. As the core region is approached the supercurrent magnitude exhibits a maximum and then decreases and the quasiparticle excitation spectrum fundamentally changes. These two effects are distinct and define two core sizes, ξ_{curr} at which $dj_S/dr = 0$ and ξ_{exc} at which the excitation spectrum changes. In a conventional superconductor $\xi_{exc} \approx \xi_{curr} = v_F / \Delta$. In the high T_c context, substantial attention has focused on ξ_{exc} (which is apparently very short³⁴) and on the possibility that the change of the excitation spectrum is not simply a collapse of the superconducting gap (as in conventional materials) but instead involves the appearance of a new form of long range order, for example antiferromagnetism or staggered flux^{35,36,37,38}. Here we wish to focus on ξ_{curr} which in a lightly doped Mott insulator may be much greater than ξ_{exc} . Writing $j_S^a = \delta F / \delta Q^a$ and using Eq 7 leads to

$$j_S^a = \rho_{S0} Q^a - \frac{Z_e^3}{8\pi v_F v_\Delta} \sum_\alpha v_\alpha^a \left(\vec{v}_\alpha \cdot \vec{Q} \right)^2 \Theta \left(\vec{v}_\alpha \cdot \vec{Q} \right) \quad (38)$$

Taking Q to be parallel to a gap node and of magnitude $1/r$ we find that the current is maximal at $\xi_{curr} = \frac{v_F^2 Z^3}{4\pi \rho_{S0} v_\Delta}$ provided that ξ_{curr} is greater than the scale over which ρ_{S0} varies. Eq 38 shows again the importance of the doping dependence of Z^e . If (as available data seem to suggest), Z^e remains constant and $\rho_{S0} \sim x$ then $\xi_{curr} \sim x^{-1}$, whereas gauge-theory based models⁴⁰ (including, we believe, those discussed in^{3,15}) lead to $Z^e \sim x$ so that ξ_{curr} is controlled by the scale dependence of ρ_S , implying $\xi_{curr} \sim x^{-1/2}$. Typical numbers for optimally doped $YBCO$ are $\rho_{S0} \sim 10 meV$, $v_F/v_\Delta \sim 15$, implying $\xi_{curr}[A] \approx 100 Z^3$, roughly consistent with muon spin rotation data¹⁶, although of course uncertainties in Z lead to large uncertainties in the numerical estimates.

The magnitude of ξ_{curr} is important because H_{c2} is essentially the field at which the vortex cores overlap, and for the resistive transition it is natural to use the 'current' definition of the vortex core size. Essentially this argument was given by Lee and Wen³ who were among the first to emphasize the importance, in the high- T_c context, of the scale over which the supercurrent varied and (on the assumption that $Z^e = 1$) concluded that $H_{c2} \sim x^2$. Future papers will examine in more detail the assumption that ξ_{curr} is the correct measure of the core size to use in estimating H_{c2} , but the plausibility of this claim may be seen for example from Eq. (16) which shows that when field becomes large enough to suppress ρ_S by a factor of order unity the intervortex spacing is of the order of ξ_{curr} .

VI. CONCLUSIONS

We have presented and compared to data a general theory of low energy properties of a d-wave superconductor. The theory has four parameters: the $T = 0$, $H = 0$ superfluid stiffness ρ_{S0} , the velocities v_F and v_Δ describing the Dirac spectrum of d-wave quasiparticles, and a quantity Z^e which expresses the coupling between quasiparticles and phase fluctuations and which we argued should be interpreted as the charge of the nodal quasiparticle. The behavior of these quantities contains information about the physics of the approach to the Mott transition, because different theoretical treatments of doped Mott insulators predict (or assume) quite different variations of these parameters with doping. The behavior of these quantities controls many aspects of the physics: in particular, the size (as defined from the supercurrent distribution) of superconducting vortices.

Two widely discussed theoretical approaches are the Brinkman-Rice-dynamical mean field theory³⁹ and the slave boson gauge theory⁴⁰. The essential ingredient of the Brinkman-Rice theory is a self-energy Σ with a strong frequency

dependence and a negligible momentum dependence. This leads to a Mott transition driven by a divergent $\partial\Sigma/\partial\omega$ implying Z^e independent of x and $v_F \sim x$. The latter prediction is in apparent contradiction to photoemission data. The essential assumption of the gauge theory approach is that current is carried by a small density of holes doped into a spin liquid environment. The fermionic excitations of the superconducting state are combinations of hole and spin-liquid states and the low density of holes leads to a small charge $Z^e \sim x$. In other words, the quasiparticles become more neutral as the Mott insulating phase is approached. One may think of this as a precursor of the 'nodal liquid' phase discussed in⁴¹. This idea also appears to be inconsistent with the available data, although, as emphasized in Section III the available data are not entirely consistent. Further, and perhaps most important, complete information is not yet available for underdoped (especially strongly underdoped) materials. We urge the experimental community to settle the issue of the data consistency, in order to finally establish the relevance of the Brinkman-Rice and gauge theory approaches to the physics of high- T_c .

Our understanding of the presently available data favors the hypothesis that Z^e and v_F remain constant as $\rho_S \rightarrow 0$. This result would appear to rule out both the Brinkman-Rice and gauge theory descriptions of the Mott physics of high T_c materials, and it is therefore interesting to understand the origin of the discrepancy. One common feature of the two approaches is that in them the Mott physics affects all of the Fermi surface in the same manner, so the reduction in charge stiffness is described by a reduction in velocity or in quasiparticle charge. If neither of these effects occurs, then the reduction in charge stiffness must be driven by a reduction in 'effective fermi surface area'. In other words, it seems likely that in high T_c materials the crucial missing ingredient is a large, doping dependent variation of the parameters around the Fermi surface so that all superfluid properties arise from condensation of fermions in a narrow and doping dependent range around the nodal direction.

Consider for example Eq. (15) which describes the reduction of the ρ_S due to depairing of the nodal quasiparticles by a non-zero superflow. A phase gradient of magnitude Q depairs electrons in an angular range $\delta\theta \sim Z^e v_F Q / v_\Delta$. In an underdoped material it seems that Z^e remains of order unity while ρ_S becomes very small. Eq. (15) then implies that exciting quasiparticles in a range $\delta\theta \ll \pi/2$ will reduce ρ_S to zero, i.e. that all or most of the supercurrent is carried by electrons small patches, of angular size $\delta\theta \sim \rho_S v_\Delta / Z^e v_F^2$ centered on the nodal points. Within this picture an interesting question is the behavior of Z for angles $\theta > \delta\theta$. Because ρ_S cannot become negative, the quasiparticles must in some manner decouple from the superfluid fluctuations (i.e Z^e must become small in these regions).

There is to our knowledge no microscopic theory of the narrow patch situation described above which is consistent with all data. One possibility is a commensurate long range order which opens a large, doping dependent gap around the antinodal points $(\pi, 0)$ which kills most of the Fermi surface leaving only hole pockets around the diagonals which then acquire a small amplitude superconducting gap. One example of this phenomenon would be the 'd-density wave' state. Another would be some form of antiferromagnetic or 'stripe' order. Two crucial consequences of such physics are a broken symmetry (which should be detectable in various spectroscopies) and a small v_Δ (determined by the observed T_c). We think that the available data do not favor this proposal. Crucially, the thermal conductivity measurements suggest that v_Δ increases when doping is decreased. An alternative which is at least qualitatively consistent with the data is preformed (d-wave) pairs which are made mostly from the electrons near $(\pi, 0)$ regions and which do not contribute to any transport. For example,⁴² proposed a theory in which a large mass in the $(\pi, 0)$ region prevented the gap maximum regions from contributing to transport. We see here that an alternative is a small Z^e . Unfortunately there is no controlled microscopic theory which yield this physics, although uncontrolled but interesting extrapolations of scaling equations have been argued to lead to this physics⁴³.

To summarize: elucidation of the experimental support (or lack thereof) for the 'patch picture' and (assuming it is relevant) clarification of its theoretical basis are two important challenges for the future.

Acknowledgements: We thank D. Bonn, P. A. Lee, M. Chiao and L. Taillefer for very helpful discussions and L. Taillefer for sharing unpublished data. We acknowledge support from NSF-DMR-00081075.

VII. APPENDIX: CURRENT DISTRIBUTION IN VORTEX LATTICE: $H_{c1} \ll H \ll H_{c2}$

A. Formalism

In the limit $H_{c2} \gg H \gg H_{c1}$ we have, for the supercurrent distribution,

$$\vec{j}_s(r) = \rho_S \vec{Q}(r) \quad (39)$$

with ρ_S the superfluid stiffness and the gradient of the superconducting phase field given by

$$\vec{Q}(r) = \sum_i \frac{\hat{z} \times (\vec{r} - \vec{r}_i)}{(\vec{r} - \vec{r}_i)^2} \quad (40)$$

The quantity appearing in the expression for the field-induced specific heat for a vortex lattice oriented at angle θ to the gap node direction is

$$C(\theta) = \frac{1}{A_V} \int dx dy |Q_x \cos(\theta) + Q_y \sin(\theta)| \quad (41)$$

where the integral is over the unit cell of the vortex lattice and A_V is the area of this cell. The result has dimension of $length^{-1}$. It is convenient to measure lengths in units of the inter-vortex spacing a and to normalize the result to the square root of the vortex density $n_V = B/\Phi_0$. Thus we write

$$C(\theta) = n_v c(\theta) \quad (42)$$

and compute $c(\theta)$ for square and triangular vortex lattices.

B. Numerical evaluation, square vortex lattice

We consider a square vortex lattice of lattice constant a , so the vortices sit at positions $na\hat{x} + ma\hat{y}$. $n_v = a^{-2}$. Eq 40 gives

$$Q_x = n_V^{1/2} \sum_{n,m} \frac{(m - y/a)}{(n - x/a)^2 + (m - y/a)^2} \quad (43)$$

$$Q_y = -Q_x(y, x) \quad (44)$$

Consider Q_x . The sum over y is most conveniently evaluated in Fourier space by writing $\sum_m \rightarrow \int du \rho(u)$ with $\rho(u) = a^{-1} \sum_m \delta(y - ma) = \sum_k e^{i2\pi ky}$. Substitution gives

$$Q_x(x, y) = \frac{\pi}{2} n_V^{1/2} \sum_n \frac{\sin(2\pi y/a)}{\sinh^2(\pi(n - x/a)) + \sin^2(\pi y/a)} \quad (45)$$

The sum on n is rapidly convergent and may easily be evaluated numerically.

We wish to evaluate Eq. 41 by integrating over the region $-a/2 < x, y < a/2$. This is most conveniently evaluated numerically by inscribing a circle in the unit cell, performing the integral over the circle in polar coordinates (to eliminate the divergence at $r \rightarrow 0$) and then integrating over the remaining regions. This latter integral is over the region $-a/2 < y < a/2$; $a/2 > |x| > \sqrt{\frac{a^2}{4} - y^2}$. We have performed this integral numerically using Mathematica; results are shown in the Table below.

C. Triangular lattice

Lattice vectors

$$\mathbf{v}_1 = a\hat{x} \quad (46)$$

$$\mathbf{v}_2 = a \left(\frac{-1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y} \right) \quad (47)$$

A general lattice vector is then $n\mathbf{v}_1 + m\mathbf{v}_2$. The unit cell is a hexagon with area $3\sqrt{3}a^2/8$.

Eq. 40 then gives, for the component of \mathbf{j} perpendicular to \mathbf{v}_1

$$Q_y = - \sum_{m,n} \frac{n - \frac{1}{2}m - x}{\left(n - \frac{1}{2}m - x\right)^2 + \left(\frac{\sqrt{3}}{2}ma - y\right)^2} \quad (48)$$

The sum over n may again be performed—it is just the previous result with $y \rightarrow x + m/2$ and $n - x/a \rightarrow \frac{\sqrt{3}}{2}ma - y$ so that

$$Q_y(x, y) = \frac{\pi}{2a} \sum_m \frac{\sin(m\pi + 2\pi x/a)}{\sinh^2\left(\pi\left(\frac{\sqrt{3}}{2}m - y/a\right)\right) + \sin^2(m\pi/2 + \pi x/a)} \quad (49)$$

Q_x is obtained by computing the component perpendicular to a different basic lattice vector and then combining appropriately.

For the Volovik effect we require $\int_{hexagon} dx dy |\mathbf{v} \cdot \mathbf{j}(x, y)|$. We find this is very well approximated by the integral over the inscribed circle. Results are shown in the Table.

Angle	c_{square}	$c_{triangle}$
0	1.5708...	1.74
$\pi/8$	1.55	1.72
$\pi/4$	1.52	1.71

Table caption: Coefficient c defined in Eq 42 for square and triangular vortex lattice as function of angle θ between lattice vector of vortex lattice and nodal direction of d-wave superconducting order parameter.

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